UDC 551.509.313/.314

OBJECTIVE ANALYSIS OF A TWO-DIMENSIONAL DATA FIELD BY THE CUBIC SPLINE TECHNIQUE

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ABSTRACT

A procedure to use the spline interpolation technique on an arbitrarily prescribed two-dimensional data field is described. For using this technique, one must obtain an initial approximation to the data at the grid points. This is achieved by fitting spherical surfaces to the data. Bidirectional spline interpolation is then applied repeatedly on the grid point estimates of the data to produce convergence to the true surface.

The spline interpolation technique and another objective analysis technique developed by Cressman are tested against an exact solution, and the resulting analyses are compared. Real temperature, geopotential height, and wind data for various pressure surfaces are analyzed by the spline method; and the results are compared to subjective analyses of the same data.

1. INTRODUCTION

With the development and continuing improvement of the electronic computer came the beginning of practical numerical weather prediction. The present day computers make it possible to use elaborate numerical models to improve forecast accuracy; however, there are certain limitations to this improvement that cannot be overcome by increasing the speed and capacity of the computer or by improving the set of equations that define the model. The two major limitations are (1) the density of the network of observation stations and (2) the unavoidable error, instrument or otherwise, introduced while making any type of measurement. The first limitation imposes a well-defined limit on the scale of atmospheric motion that can be resolved by the model. Of course, the solution to this problem is simply to increase the density of the observing network, thereby obtaining the "scale of data" necessary for predicting the corresponding scale of motion. The second limitation is far more difficult to eliminate. Any set of observations contains certain random errors as well as some small-scale fluctuations (noise). In using this data in any numerical model, it is of utmost importance to eliminate these errors and the noise if the final results are to be meaningful. This has been demonstrated, for example, by the studies of Best (1956) and Berggren (1957). Since meteorological prediction is an initial value problem, it becomes imperative to begin any numerical weather prediction with the "best possible" representation of the real data.

Various attempts at solving this problem have been introduced in meteorology within the past 15 yr; these will be referred to as "objective analysis." An appropriate definition of objective analysis was given by Gandin (1965):

"Objective analysis includes the development and realization of methods which make it possible to use the measurement data of meteorological stations to reconstruct objectively the fields of the meteorological elements (variable), or at any rate to specify their values at the nodes (grid points) of some type of regular network."

Actually, objective analysis includes three distinct functions: (1) elimination or correction of gross errors in the data field, (2) interpolation of data to obtain values on a grid, and (3) smoothing of the resulting values at the grid points.

Probably, one of the first attempts at objective analysis of meteorological data was by Panofsky (1949). Panofsky represented a field by cubic polynomials and showed that the introduction of random observational errors into the data field resulted in only minor variations in the polynomial representation.

By 1954, the need for a better objective-analysis technique to be used in conjunction with the rapidly developing field of numerical weather prediction resulted in the method developed by Gilchrist and Cressman (1954). Their method was based on fitting a second-degree polynomial by the method of least squares to the data in a limited area around each grid point. Wind values were incorporated in the scheme by using the geostrophic assumption to determine ∇h (where h is the deviation of height from the standard atmosphere) at a data point. Thus, each data point supplied three pieces of information, h, $\partial h/\partial x$, and $\partial h/\partial y$ to be used in the least-squares fitting. For regions of sparse data, this method proved to be inadequate since at least six initial pieces of information were needed to determine the second-degree polynomials that defined the field. In fact, Gilchrist and Cressman found that, with less than 10 pieces of data, the calculation was subject to significant error. In regions of sufficient data, however, they found that numerical predictions based on the objective analyses were an improvement over those based on subjective analyses.

About a year after Gilchrist and Cressman (1954) introduced their objective analysis technique, Bergthórsson and Döös (1955) developed a new approach to objective analysis. Their method differed from Gilchrist and Cressman's in that they first determined what is now called the "preliminary field." This field was computed from the weighted mean of the forecast values at the grid points and the climatological norms for that time. The data obtained from the observation stations were then used in a sequence of three corrections that were then applied to the preliminary field with different weights, depending on the distance from station to grid point. The numerical predictions based on the objective analysis technique of Bergthórsson and Döös were approximately the same as the predictions based on the subjective analyses.

Approximately 5 yr after the development of his first objective-analysis technique, Cressman (1959) introduced a modification of the Bergthórsson-Döös (1955) method. In his new method, Cressman used a preliminary field that was usually the forecast for the time of the observation data. Weighted corrections based on the new observations were then applied to the preliminary field. The corrections were defined as a function of the distance (d) from grid point to station. The weight factor (W) for each correction is given by

$$W=\frac{n^2-d^2}{n^2+d^2} \text{ for } d\leq n$$
$$=0 \qquad \text{for } d>n$$

where n is a multiple of N, the grid interval. The correction procedure was then repeated for decreasing multiples of N. Cressman's new method of objective analysis resulted in better numerical predictions than those based upon his previous method of analysis.

Improvements of existing techniques have been developed, most of which are based on the inclusion of additional information such as surface data, vorticity, geostrophic approximation, etc. (Döös and Eaton 1957, Johnson 1957, Sasaki 1958, Aubert 1959, Masuda and Arakawa 1962, and Teweles and Snidero 1962).

Although the above objective analyses give satisfactory results for regions of sufficiently dense observation stations, a reliable technique that will operate satisfactorily over regions of sparse data remains to be developed. An attempt to develop such a technique is presented in the following sections.

2. BASIC SPLINE THEORY

The problem of passing a smooth curve through a given set of points (N) has been solved mechanically by using a thin elastic strip to define the curve (fig. 1A). This strip is commonly called a spline. Although a polynomial of degree $N-1$ could be determined that would also pass through the same set of points, the curve defined by the spline will be smoother. From the theory of elasticity, it can be shown that a spline will have the minimum possible strain energy (Love 1944, Holladay 1957), that is,

$$\int K^2 ds = \text{minimum}$$

where K is the curvature and ds is the arc length. Since strain energy is a measure of the smoothness of a curve, the spline *must* define the smoothest curve for a given set of points.

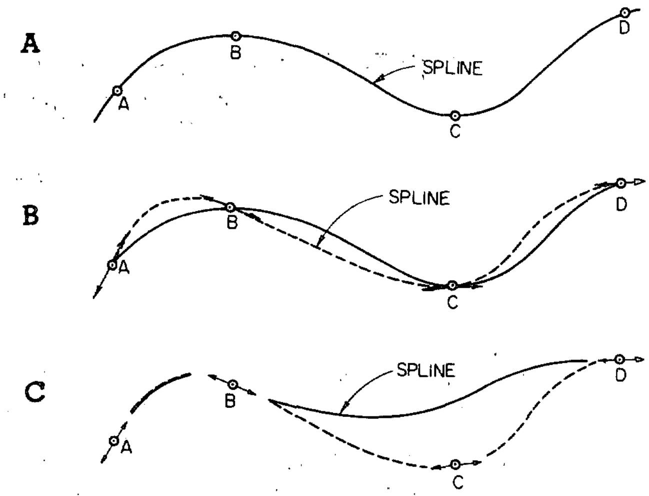

FIGURE 1.—(A) spline passing through points A, B, C, and D; (B) spline passing through A, B, C, and D where slopes have been prescribed at the points; (C) spline curve that results when one of the constraints (point C) has been removed.

If, in addition to knowing the points through which the spline passes, the slopes are prescribed at these points, the spline will define a new curve (fig. 1B) having a minimum of strain energy for the new set of constraints.

The concept of obtaining a smooth curve by passing a spline through a given set of points may be applied to the analysis of data. For this purpose, the spline curve between any two data points will have to be approximated mathematically by a polynomial representation. These piecewise polynomials will then have to be joined together under certain specified constraints like the continuity of the function and its derivatives at the data points. These constraints, of course, depend on the nature of the phenomenon under investigation. Fowler and Wilson (1966) have developed a method of determining a series of cubic equations that, when spliced together with continuous slope and curvature at the junction points, approximate a spline for a given set of data and slopes. Pearce and Riehl (1968) used piecewise quadratic polynomials to allow for the sharp but real changes that occur in representing a vertical wind profile.

Normally, it is not desirable to fit a curve exactly through all the data points; and so, some type of smoothing is performed on the data. Now, it is interesting to note that, if one of the constraints in figure 1B is removed, the resulting position of the spline (fig. 1C) is one of even less strain energy (i.e., the spline defines a smoother curve). Thus, an equation that approximates the new spline should give a smoother representation of the data. However, since it is not desirable to completely eliminate any data points, the new spline could instead be used to determine the magnitude and direction of the movement of a data point such that "controlled" smoothing may be done. Fowler and Wilson (1962) made use of this idea to smooth their series of continuous cubic equations.

3. MATHEMATICAL APPROXIMATION TO SPLINES

The mathematical development of the more widely used spline approximation, the cubic spline, is described below. A cubic spline fitting requires a general third-degree polynomial between any two points of a given set of data. Hence, 10 inital conditions must be known. If the coordinate system is translated and rotated so that the first point is at the origin and the second is on the x' axis, the general equation for a third-degree polynomial reduces to

$$y'=Ax'^3+Bx'^2+Cx'+D \tag{1}$$

where the primes indicate the new coordinate axes. The solution to this new system requires only four initial conditions. Assuming that the coordinates of the two points (end points) are known, one must still determine two additional conditions. These are obtained in the following manner. Consider figure 2. A circle is fitted to the set of points defined by end point B and the two adjacent points A and C. The derivative of the resulting equation may then be solved for the slope at B. Similarly, the slope at E is determined using points D, E, and F. Then, the four initial conditions are:

(at the first end point) $x'=y'=0$ (2)

slope$=S1'$ (3)

and

(at the second end point) $x'=d,\ y'=0$ (4)

slope$=S2'$ (5)

where d is the distance between the end points.

The equation for the slope is given by the first derivative of eq (1):

$$\frac{dy'}{dx'}=3Ax'^2+2Bx'+C. \tag{6}$$

The system consisting of eq (1) through (6) may now be solved in terms of $S1'$, $S2'$, and d. The resulting expression for y', then, is

$$y'=\frac{(S1'+S2')}{d^2}x'^3-\frac{(2S1'+S2')}{d}x'^2+S1'x'. \tag{7}$$

Due to the rotation of the coordinate system, the end-point slopes in the original system $S1$ and $S2$ have been transformed to new values $S1'$ and $S2'$ in the primed

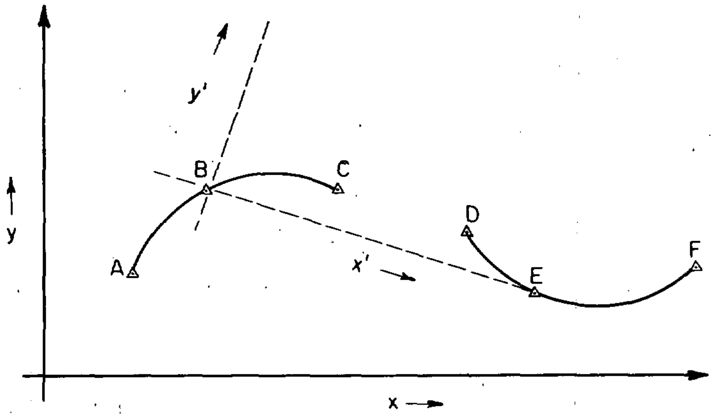

FIGURE 2.—Determination of slopes at data points by fitting circles to the data.

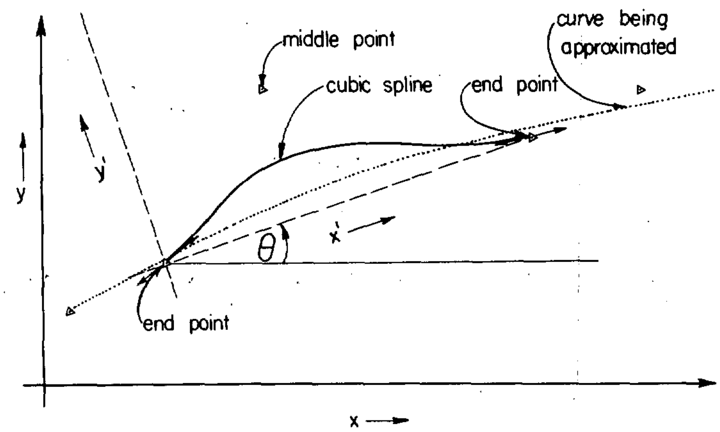

FIGURE 3.—Normal position assumed by the cubic spline when fitted to the end points of a 3-data interval.

system. This transformation is given by

$$S1'=\frac{S1-TR}{1+S1\cdot TR}$$

and

$$S2'=\frac{S2-TR}{1+S2\cdot TR}$$

where TR is the tangent of the angle of rotation (θ).

Now consider a data interval to be the interval defined by the end points of three successive data points. A cubic equation fitted to the end points of each successive interval for a given set of data will usually pass between the curve to be approximated and the middle data point of the interval (fig. 3). This suggests an iterative smoothing procedure resulting in the convergence of the data toward the curve.

Starting with the first interval, one constructs a cubic for that interval. The middle data point is then adjusted toward the cubic. Since the location of the curve to be approximated is not known in most cases, the middle point is only moved some fraction of the total distance between

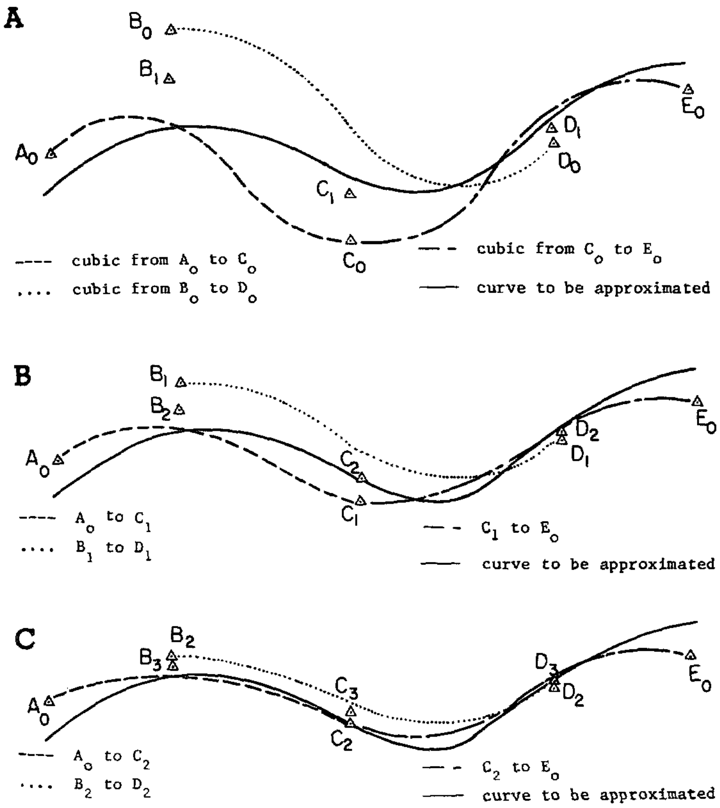

FIGURE 4.—(A) set of three cubic splines approximating a curve; (B) second iteration in the cubic spline convergence; (C) third iteration in the cubic spline convergence.

the point and the cubic curve. The procedure is repeated in the succeeding intervals. Figure 4A illustrates this procedure where the adjusted points are indicated by the higher subscripts. For example, a cubic is fitted to A_0 and C_0. The middle point B_0 is then moved toward the cubic to its adjusted position B_1. In the next interval, the cubic is fitted to B_0 and D_0, which adjusts C_0 to C_1. Similarly, D_0 is adjusted to D_1 in the interval (C_0, E_0). Upon completion of the data adjustment for all intervals, the entire process is repeated until satisfactory convergence is achieved or the benefit of additional smoothings becomes impractical. Figures 4B and 4C depict the second and third iterations in a series of three smoothing iterations for a set of five data points. In general, spline interpolation converges rapidly as shown by Ahlberg and Nilson (1963) and has the additional property of being insensitive to round-off errors.

4. TWO-DIMENSIONAL CONSIDERATIONS

The results of approximating curves by splines have been so successful (Curtis and Powell 1966; Walsh et al. 1962) that it would seem desirable to extend the theory to approximate surfaces. Birkhoff and DeBoor (1965) developed a method using bicubic spline interpolation that approximates surfaces on a rectangular field given the data value (u) at all grid points, the normal derivative ($\partial u/\partial n$) at the boundary grid points of each elemental rectangle, and the cross derivatives ($\partial^2 u/\partial x \partial y$) at the four corners of the field. In practice, however, most data fields are not known at a regular grid network; and the problem is therefore to interpolate the known data to the grid points and smooth the resulting fields.

Since circles were used successfully by Fowler and Wilson (1962) to obtain an approximation to the slope at each point in their curve-fitting routine, it would seem logical to fit spheres to the data to approximate the surface and slope in a surface-fitting routine. Since the value of the surface is desired only at the grid points, the problem may be reduced to one of splicing the surfaces of the spheres together at a discrete number of points.

Given an arbitrarily located set of two-dimensional data points, one places the points in an order according to increasing values of the x coordinate. A grid network is defined (grid length is arbitrary) to cover the data field such that all data points fall within the network. Since, for a given grid point, not all the data influence the value at that point, what shall be referred to as the "band of influence" is defined for each grid line (y=constant). The vertical plane coincident with the grid line is called the grid plane.

For a given grid line, those data points that fall into the band of influence are used to define the spheres. The surface value at each grid point on the grid line can then be determined by substituting the coordinate of the grid point into the equation for the appropriate sphere. The equation for a sphere is given by

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2. \tag{8}$$

Given four data points, one may solve eq (8) for a, b, and c (the coordinates of the center of the sphere) and r (the radius of the sphere). For a given grid line $y=y_j$ that passes through the sphere and for grid point $x=x_i$ in the domain of the sphere, eq (8) may be solved for the surface $z_{i,j}$ at the grid point.

A detailed description of the procedure for determining an approximation to the surface at all grid points along with an error check for these approximations is given by Fritsch (1969).

Starting with the first grid line y_1, one smooths each grid line, using the cubic spline routine as outlined in section 3. Upon completion of smoothing in the x direction, the entire field is smoothed in the same manner in the y direction. Repeated smoothings in the x and y directions eliminate directional bias; and after three or four of these bidirectional smoothings, the grid point values converge to a surface.

5. RESULTS

For properly testing the spline method, one would have an advantage knowing an exact solution for the surface. Thus, in addition to a qualitative evaluation of the results,

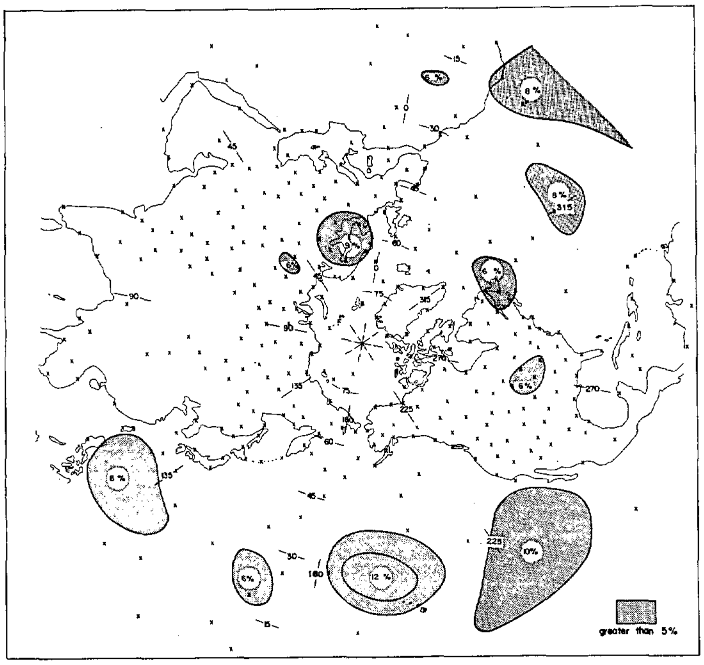

FIGURE 5.—Data-point locations (marked by crosses) and the percent error for the spline approximation to the exact solution. The unshaded areas have less than 5-percent error; the percent error shown in each shaded area is the maximum error for that area.

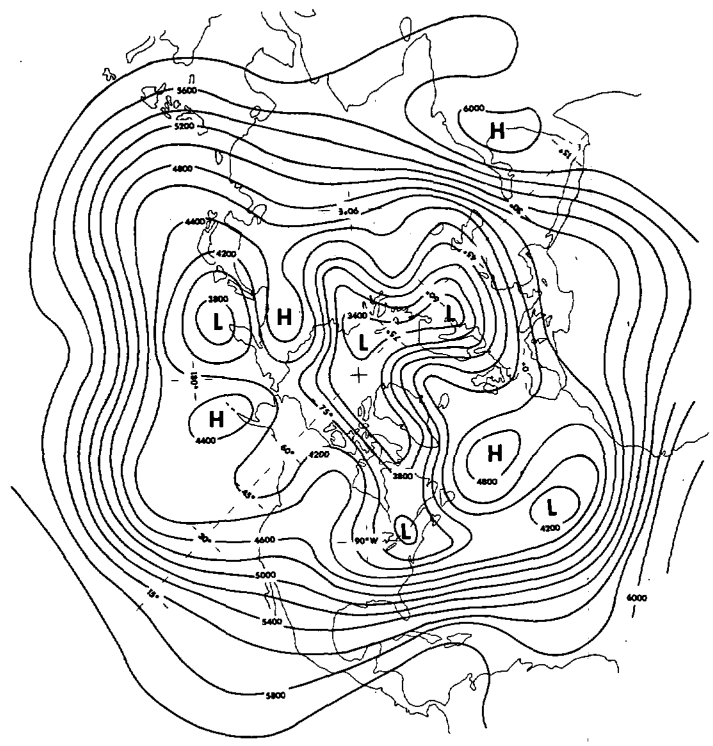

FIGURE 6.—Exact solution for the stream function for the upper level of a two-layer baroclinic spectral general circulation model; units of height, meters.

a quantitative measure of the error could be determined. This would have the additional advantage of being able to compare, quantitatively, the error of the spline objective analysis to the error of other methods.

Based on these considerations, a test was constructed using a solution for the stream function for the upper level of a two-layer baroclinic spectral general circulation model. Orographic and land-sea effects were not included in the test model. The solution selected was for the Northern Hemisphere and (when multiplied by the appropriate conversion factor) could be interpreted as the height of the 250-mb pressure surface. A total of 356 data points was selected, most of which are established radiosonde stations (fig. 5). An 18×72 grid (5° latitude grid length) was used, and the height at all grid points was determined from the exact solution (fig. 6). In figure 5, the percent error at each grid point is defined as

$$\% \text{ error} = 100\% \times (\text{exact height minus spline analysis height})/(\text{exact height}).$$

The results of applying the spline technique to the data at the radiosonde locations are shown in figure 7. Another objective analysis technique based on that of Cressman (1959) was applied to the same data, and the results are shown in figure 8. Application of Cressman's technique normally requires a preliminary field. This field is usually the forecast for the observation time of the data being analyzed. Since a forecast, other than the exact solution, was not available to the author (Fritsch), the mean value

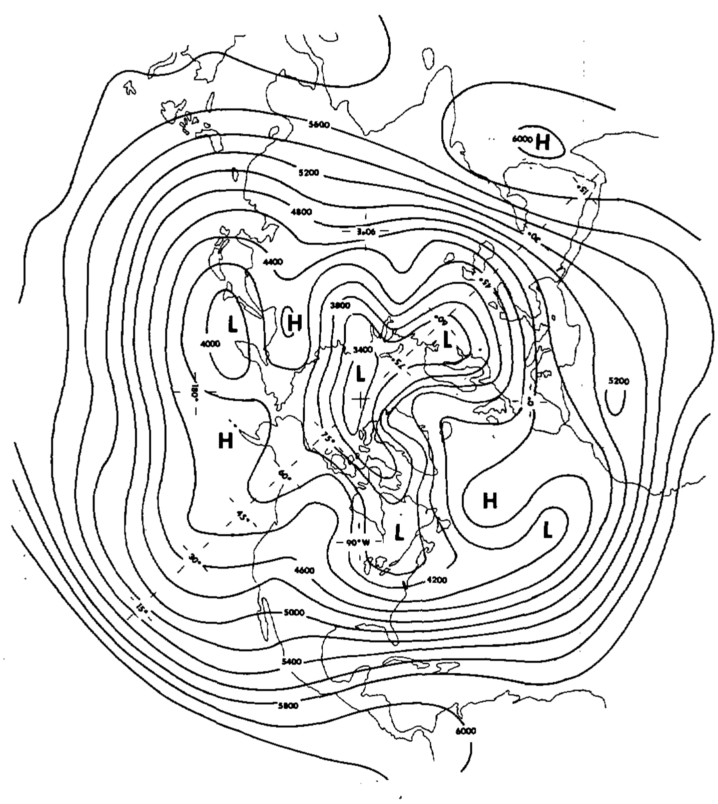

FIGURE 7.—Spline approximation to the exact solution; the standard error is 147 m.

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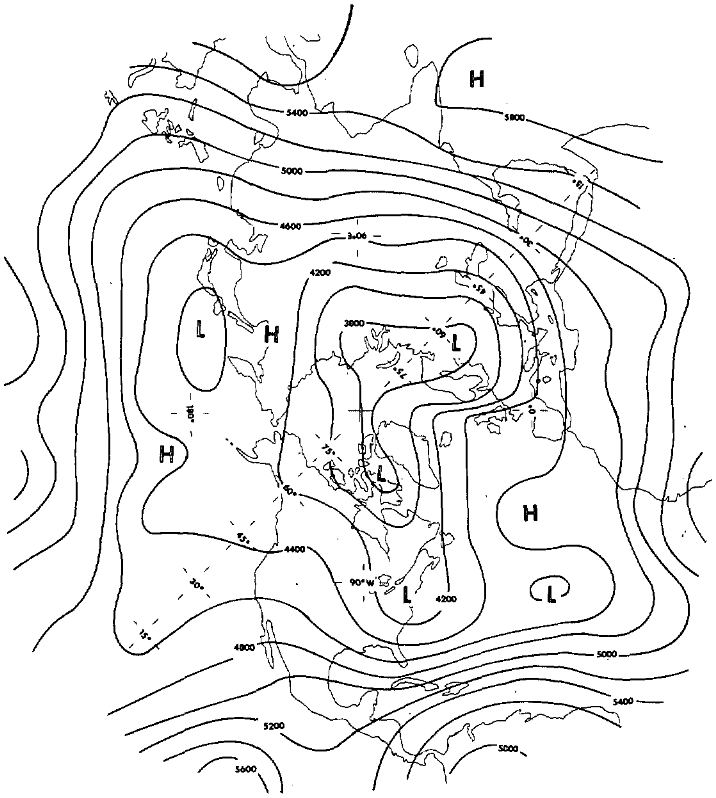

FIGURE 8.—Cressman's approximation to the exact solution; the standard error is 321 m.

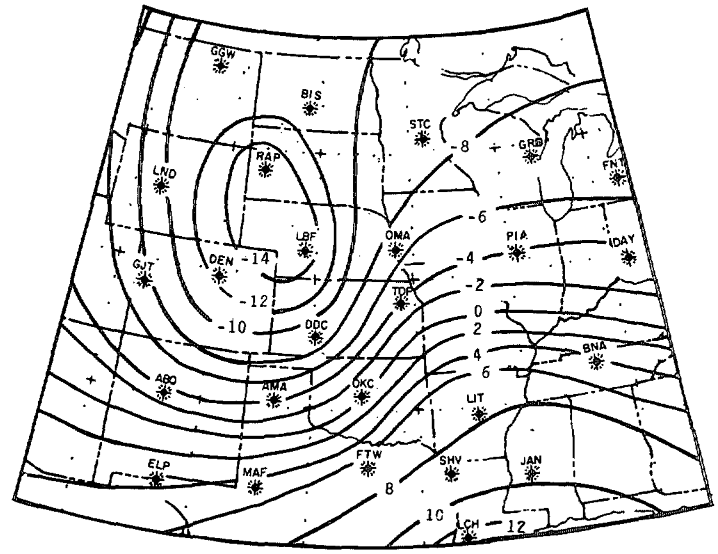

FIGURE 9.—Spline analysis of the 850-mb temperature (°C) for the Central United States at 0000 GMT on Dec. 28, 1966.

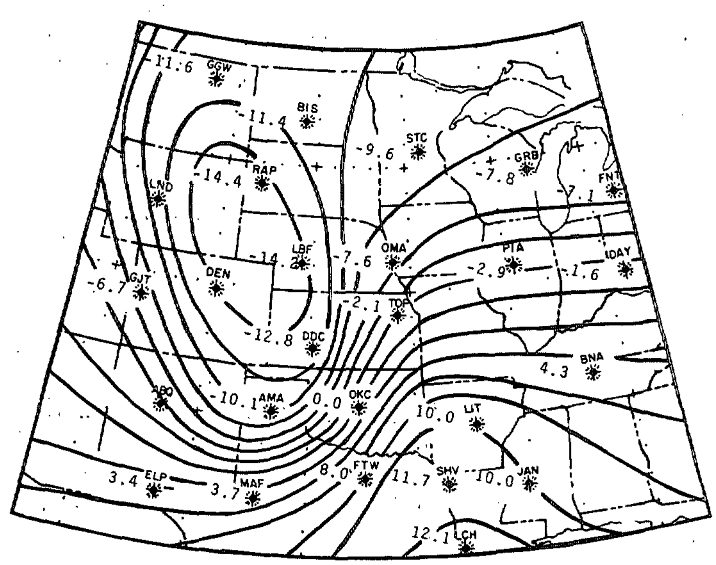

FIGURE 10.—Subjective analysis of the 850-mb temperatures (°C) for the Central United States at 0000 GMT on Dec. 28, 1966.

of the exact solution along each line of constant latitude (every 5°) was substituted at all grid points along the respective latitude line. For obtaining the preliminary field at observation points, a linear interpolation was applied between the mean heights of the latitude lines immediately north and south of each observation point. Cressman's technique also uses the geostrophic approximation on the observed winds to correct the height field. Since the exact solution used to test the Cressman and spline techniques was in geostrophic balance, the geostrophic wind was calculated at each observation point and subsequently used in the Cressman technique.

Other than the mean height approximation for the preliminary field and the geostrophic wind in place of the observed wind, the Cressman technique used in the test comparison was essentially the same (scan radii, smoothing, etc.) as that discussed in Cressman (1959). The standard error was computed for both techniques with the result that the error for Cressman's technique was on the order of twice the error of the spline technique. The percent error for the spline solution was calculated at each grid point and is shown in figure 5. The only region where greater than 10-percent error occurs is located over that section of the Pacific Ocean where there is an extreme lack of data (fig. 5). The mean error for the entire Northern Hemisphere was approximately 3 percent; and in regions of good data coverage, this figure usually dropped to 0–2 percent.

In practical weather analysis, quite frequently it becomes necessary to analyze pressure gradient and/or temperature discontinuities (fronts). Since one of the functions of most objective analyses is to smooth the data, the discontinuity is often lost completely or is smoothed to such an extent that its frontal characteristics are no longer significant. Since it is usually desirable to maintain frontal characteristics, a test was constructed to determine how the cubic spline technique performs on discontinuities. The temperature data for the United States at 0000 GMT on Dec. 28, 1966, was selected for analysis. On this date. a particularly well-defined cold front was located near the middle of the country. Figure 9, the spline analysis of

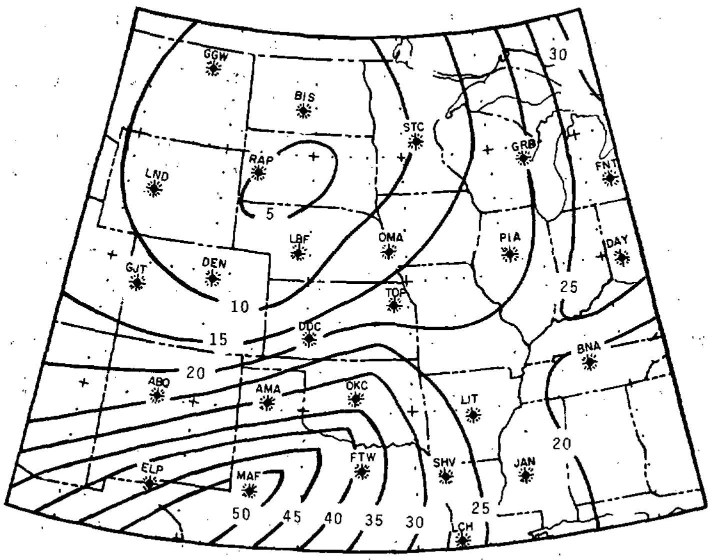

FIGURE 11.—Spline analysis of the 500-mb winds (kt) for the Central United States at 0000 GMT on Dec. 28, 1966.

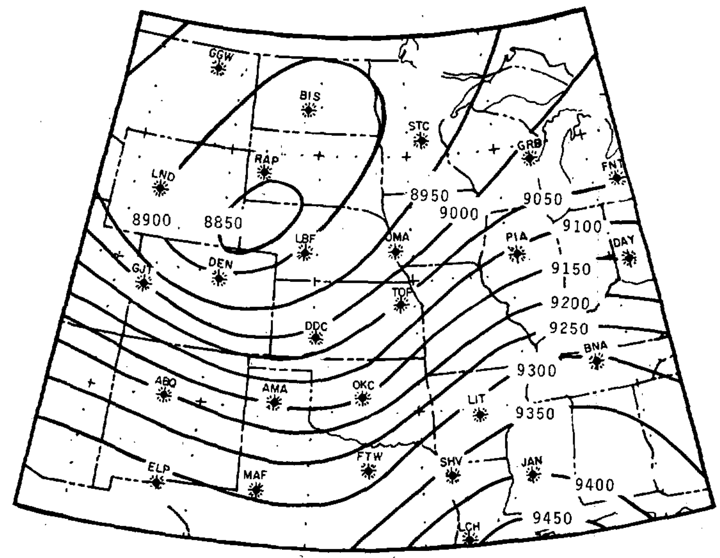

FIGURE 13.—Spline analysis of the 300-mb geopotential heights (m) for the Central United States at 0000 GMT on Dec. 28, 1966.

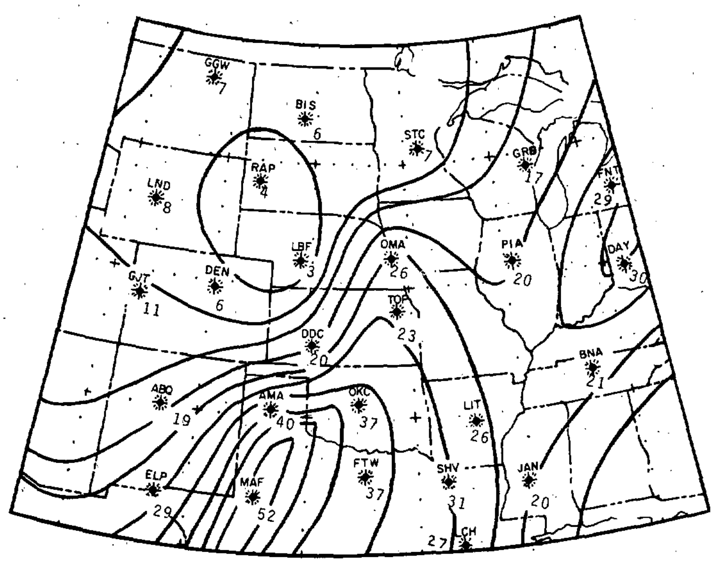

FIGURE 12.—Subjective analysis of the 500-mb winds (kt) for the Central United States at 0000 GMT on Dec. 28, 1966.

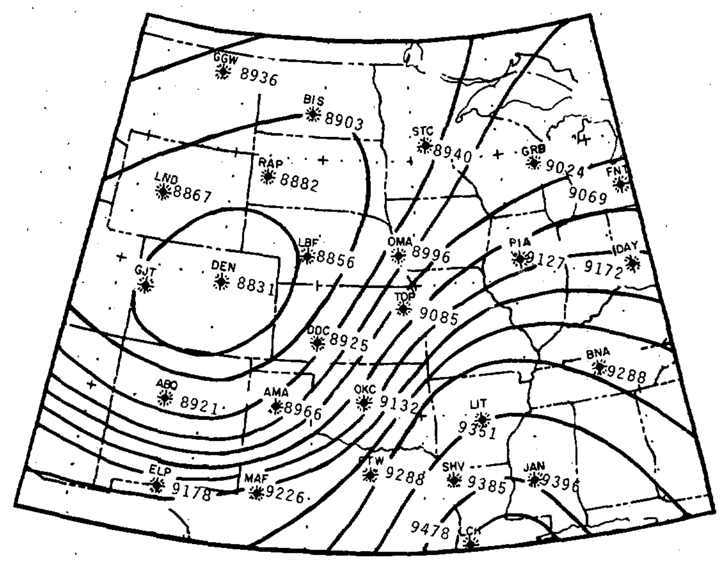

FIGURE 14.—Subjective analysis of the 300-mb geopotential heights (m) for the Central United States at 0000 GMT on Dec. 28, 1966.

the front at the 850-mb pressure surface where the front is most intense, indicates that the loss of frontal characteristics appears to be minimal. Figure 10 is the corresponding subjective analysis. A comparison of objective and subjective analyses shows good agreement both along and across the front. It is possible that the frontal characteristics may be even better resolved if a quadratic spline is used (e.g., Pearce 1968).

For the same time and location as the temperature data, the wind and height data were also analyzed objectively (figs. 11 and 13, respectively) and subjectively (figs. 12 and 14, respectively). Additional spline analyses of real temperature, wind, and height fields are shown in Fritsch (1969).

6. CONCLUSIONS

Objective analysis by the spline technique appears to be a satisfactory method for two-dimensional data analysis. Analysis of regions with poor data coverage also appears to give satisfactory results except in those situations where the features being analyzed are defined by less than three pieces of data. The magnitude of the gradient of the data to be analyzed does not seem to have any undesirable effects on the performance of the technique, provided that the input parameters are properly defined.

ACKNOWLEDGMENTS

The author wishes to express his sincere appreciation to Dr. D. B. Rao for his many hours of helpful discussion and encouragement, to Dr. Ferdinand Baer for his constructive comments and suggestions, to Mr. Fred Alyea for providing the general circulation model without which a quantitative evaluation of the results would not have been possible, and to Mr. Norval McMillin for his aid and suggestions in programming the analysis procedure.

REFERENCES

Ahlberg, J. H., and Nilson, E. N., "Convergence Properties of the Spline Fit," *Journal of the Society for Industrial and Applied Mathematics*, Vol. 11, No. 3, Philadelphia, Pa., Mar. **1963**, pp. 95–104.

Aubert, Eugene J., "Objective Map Analysis Utilizing the Concept of Circulation and Vorticity," *Journal of Meteorology*, Vol. 16, No. 4, Aug. **1959**, pp. 427–435.

Berggren, Roy K., "On the Accuracy of 500 Mb Analysis With Special Reference to Numerical Forecasting," *Tellus*, Vol. 9, No. 3, Stockholm, Sweden, Aug. **1957**, pp. 323–340.

Bergthórsson, Páll, and Döös, Bo R., "Numerical Weather Map Analysis," *Tellus*, Vol. 7, No. 3, Stockholm, Sweden, Aug. **1955**, pp. 329–340.

Best, William H., "Differences in Numerical Prognoses Resulting From Differences in Analyses," *Tellus*, Vol. 8, No. 3, Stockholm, Sweden, Aug. **1956**, pp. 351–356.

Birkhoff, Garrett, and Deboor, C. R., "Piecewise Polynomial Interpolation and Approximation," *Symposium on Approximation of Functions, General Motors Research Laboratories, Warren, Michigan, August 31–September 2, 1964*, Elsevier Publishing Co., New York, N.Y., **1965**, pp. 164–190.

Cressman, George P., "An Operational Objective Analysis System," *Monthly Weather Review*, Vol. 87, No. 10, Oct. **1959**, pp. 367–374.

Curtis, A. R., and Powell, M. J. D., "Using Cubic Splines to Approximate Functions of One Variable to Prescribe Accuracy," *Technical Paper* 252, Contract No. HL66/6251, Mathematical Branch, Atomic Energy Research Establishment, Harwell, Berkshire, England, **1966**, 14 pp.

Döös, Bo R., and Eaton, M. A., "Upper-Air Analysis Over Ocean Areas," *Tellus*, Vol. 9, No. 2, Stockholm, Sweden, May **1957**, pp. 184–194.

Fowler, A. H., and Wilson, C. W., "Cubic Spline, a Curve Fitting Routine," *Report* No. Y-1400, Contract No. W-7405-ENG-26, Nuclear Division, Union Carbide Corp., Oak Ridge, Tenn., **1962**, pp. 1–41.

Fritsch, J. Michael, "Objective Analysis of a Two Dimensional Data Field by the Cubic Spline Technique," *Atmospheric Science Paper* No. 143, Department of Atmospheric Science, Colorado State University, Ft. Collins, Aug. **1969**, 34 pp.

Gandin, Lev Semenovich, *Objective Analysis of Meteorological Fields*, Israel Program for Scientific Translations, Jerusalem, **1965**, 242 pp. (translation of *Ob''ektivnyĭ Analiz Meterologicheskikh Poleĭ*, Gidrometeozdat, Leningrad, U.S.S.R., 1963, 286 pp.

Gilchrist, Bruce, and Cressman, George P., "An Experiment in Objective Analysis," *Tellus*, Vol. 6, No. 4, Stockholm, Sweden, Nov. **1954**, pp. 309–318.

Holladay, John C., "Smoothest Curve Approximation," *Mathematical Tables and Other Aids to Computation*, Vol. 11, No. 60, National Research Council, Washington, D.C., Oct. **1957**, pp. 233–243.

Johnson, D. H., "Preliminary Research in Objective Analysis," *Tellus*, Vol. 9, No. 3, Stockholm, Sweden, Aug. **1957**, pp. 316–322.

Love, August Edward Hough, A Treatise on *the Mathematical Theory of Elasticity*, 4th Edition, Cambridge University Press, London, England, **1944**, 643 pp. (see pp. 59–73).

Masuda, Yoshinobu, and Arakawa, Akio, "On the Objective Analysis for Surface and Upper Level Maps," *Proceedings of the International Symposium on Numerical Weather Prediction, Tokyo, Japan, November 7–13, 1960*, Meteorological Society of Japan, Tokyo, Mar. **1962**, pp. 55–66.

Panofsky, Hans A., "Objective Weather-Map Analysis," *Journal of Meteorology*, Vol. 6, No. 6, Dec. **1949**, pp. 386–392.

Pearce, Robert P., and Riehl, Herbert, "Studies on Interaction Between Synoptic and Mesoscale Weather Elements in the Tropics," *Atmospheric Science Paper* No. 126, Department of Atmospheric Science, Colorado State University, Ft. Collins, June **1968**, pp. II–4—II–6.

Sasaki, Yoshikazu, "An Objective Analysis Based on the Variational Method," *Journal of the Meteorological Society of Japan*, Ser. 2, Vol. 36, No. 3, Tokyo, June **1958**, pp. 77–88.

Teweles, Sidney, and Snidero, Mirco, "Some Problems of Numerical Objective Analysis of Stratospheric Constant Pressure Surfaces," *Monthly Weather Review*, Vol. 90, No. 4, Apr. **1962**, pp. 147–155.

Walsh, J. L., Ahlberg, J. H., and Nilson, E. N., "Best Approximation Properties of the Spline Fit," *Journal of Mathematics and Mechanics*, Vol. 11, No. 2, Graduate Institute for Mathematics and Mechanics, Indiana University, Bloomington, Ind.,. Mar **1962**, pp. 225–234.

[*Received April 7, 1970; revised September 8, 1970*]